



## Solutions Pamphlet

American Mathematics Competitions

65<sup>th</sup> Annual

# AMC 12 B

American Mathematics Contest 12 B  
Wednesday, February 19, 2014

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, email, internet or media of any type during this period is a violation of the competition rules.*

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1. **Answer (C):** Leah has 7 pennies and 6 nickels, which are worth 37 cents.
2. **Answer (C):** The special allows Orvin to purchase balloons at  $\frac{1+\frac{3}{4}}{2} = \frac{5}{6}$  times the regular price. Because Orvin had just enough money to purchase 30 balloons at the regular price, he may now purchase  $30 \cdot \frac{6}{5} = 36$  balloons.
3. **Answer (E):** The fraction of Randy's trip driven on pavement was  $1 - \frac{1}{3} - \frac{1}{5} = \frac{7}{15}$ . Therefore the entire trip was  $20 \div \frac{7}{15} = \frac{300}{7}$  miles.
4. **Answer (B):** Let a muffin cost  $m$  dollars and a banana cost  $b$  dollars. Then  $2(4m + 3b) = 2m + 16b$ , and simplifying gives  $m = \frac{5}{3}b$ .
5. **Answer (A):** Denote the height of a pane by  $5x$  and the width by  $2x$ . Then the square window has height  $2 \cdot 5x + 6$  inches and width  $4 \cdot 2x + 10$  inches. Solving  $2 \cdot 5x + 6 = 4 \cdot 2x + 10$  gives  $x = 2$ . The side length of the square window is 26 inches.
6. **Answer (D):** Let  $a$  be the amount in a regular lemonade. Then a large lemonade holds  $\frac{3}{2}a$ , and Ann had  $\frac{1}{4} \cdot \frac{3}{2}a = \frac{3}{8}a$  lemonade left right before she gave Ed part of her drink. She gave him  $\frac{1}{3} \cdot \frac{3}{8}a + 2 = \frac{1}{8}a + 2$  ounces. Because Ann and Ed drank the same amount of lemonade, it follows that  $a + (\frac{1}{8}a + 2) = \frac{3}{2}a - (\frac{1}{8}a + 2)$ , and  $4 = \frac{1}{4}a$ . Thus  $a = 16$  ounces,  $\frac{3}{2}a = 24$  ounces, and together they drank  $16 + 24 = 40$  ounces.
7. **Answer (D):** Let  $x = \frac{n}{30-n}$  so that  $n = \frac{30x}{x+1}$ . Because  $x$  and  $x+1$  are relatively prime, it follows that  $x+1$  must be a factor of 30. Because  $n$  is positive and less than 30 it follows that  $x+1 \geq 2$ . Thus  $x+1$  equals 2, 3, 5, 6, 10, 15, or 30. Hence there are 7 possible values for  $n$ , namely 15, 20, 24, 25, 27, 28, and 29.
8. **Answer (C):** As indicated by the leftmost column  $A+B \leq 9$ . Then both the second and fourth columns show that  $C=0$ . Because  $A$ ,  $B$ , and  $C$  are distinct digits,  $D$  must be at least 3. The following values for  $(A, B, C, D)$  show that  $D$  may be any of the 7 digits that are at least 3:  $(1, 2, 0, 3)$ ,  $(1, 3, 0, 4)$ ,  $(2, 3, 0, 5)$ ,  $(2, 4, 0, 6)$ ,  $(2, 5, 0, 7)$ ,  $(2, 6, 0, 8)$ ,  $(2, 7, 0, 9)$ .

9. **Answer (B):** By the Pythagorean Theorem,  $AC = 5$ . Because  $5^2 + 12^2 = 13^2$ , the converse of the Pythagorean Theorem applied to  $\triangle DAC$  implies that  $\angle DAC = 90^\circ$ . The area of  $\triangle ABC$  is 6 and the area of  $\triangle DAC$  is 30. Thus the area of the quadrilateral is  $6 + 30 = 36$ .
10. **Answer (D):** Let  $m$  be the total mileage of the trip. Then  $m$  must be a multiple of 55. Also, because  $m = cba - abc = 99(c - a)$ , it is a multiple of 9. Therefore  $m$  is a multiple of 495. Because  $m$  is at most a 3-digit number and  $a$  is not equal to 0,  $m = 495$ . Therefore  $c - a = 5$ . Because  $a + b + c \leq 7$ , the only possible  $abc$  is 106, so  $a^2 + b^2 + c^2 = 1 + 0 + 36 = 37$ .

**OR**

Let  $m$  be the total mileage of the trip. Then  $m$  must be a multiple of 55. Also, because  $m = cba - abc = 99(c - a)$ ,  $c - a$  is a multiple of 5. Because  $a \geq 1$  and  $a + b + c \leq 7$ , it follows that  $c = 6$  and  $a = 1$ . Therefore  $b = 0$ , so  $a^2 + b^2 + c^2 = 37$ .

11. **Answer (E):** The numbers in the list have a sum of  $11 \cdot 10 = 110$ . The value of the 11th number is maximized when the sum of the first ten numbers is minimized subject to the following conditions.

- If the numbers are arranged in nondecreasing order, the sixth number is 9.
- The number 8 occurs either 2, 3, 4, or 5 times, and all other numbers occur fewer times.

If 8 occurs 5 times, the smallest possible sum of the first 10 numbers is

$$8 + 8 + 8 + 8 + 8 + 9 + 9 + 9 + 9 + 10 = 86.$$

If 8 occurs 4 times, the smallest possible sum of the first 10 numbers is

$$1 + 8 + 8 + 8 + 8 + 9 + 9 + 9 + 10 + 10 = 80.$$

If 8 occurs 3 times, the smallest possible sum of the first 10 numbers is

$$1 + 1 + 8 + 8 + 8 + 9 + 9 + 10 + 10 + 11 = 75.$$

If 8 occurs 2 times, the smallest possible sum of the first 10 numbers is

$$1 + 2 + 3 + 8 + 8 + 9 + 10 + 11 + 12 + 13 = 77.$$

Thus the largest possible value of the 11th number is  $110 - 75 = 35$ .

12. **Answer (B):** Denote a triangle by the string of its side lengths written in nonincreasing order. Then  $S$  has at most one equilateral triangle and at most one of the two triangles 442 and 221. The other possible elements of  $S$  are 443, 441, 433, 432, 332, 331, and 322. All other strings are excluded by the triangle inequality. Therefore  $S$  has at most 9 elements.

13. **Answer (C):** There is a triangle with side lengths 1,  $a$ , and  $b$  if and only if  $a > b - 1$ . There is a triangle with side lengths  $\frac{1}{b}$ ,  $\frac{1}{a}$ , and 1 if and only if  $\frac{1}{a} > 1 - \frac{1}{b}$ , that is,  $a < \frac{b}{b-1}$ . Therefore there are no such triangles if and only if  $b - 1 \geq a \geq \frac{b}{b-1}$ . The smallest possible value of  $b$  satisfies  $b - 1 = \frac{b}{b-1}$ , or  $b^2 - 3b + 1 = 0$ . The solution with  $b > 1$  is  $\frac{1}{2}(3 + \sqrt{5})$ . The corresponding value of  $a$  is  $\frac{1}{2}(1 + \sqrt{5})$ .

14. **Answer (D):** Denote the edge lengths by  $x$ ,  $y$ , and  $z$ . The surface area is  $2(xy + yz + zx) = 94$  and the sum of the lengths of the edges is  $4(x + y + z) = 48$ . Therefore  $144 = (x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx) = x^2 + y^2 + z^2 + 94$ , so  $x^2 + y^2 + z^2 = 50$ . By the Pythagorean Theorem applied twice, each of the 4 internal diagonals has length  $\sqrt{50}$ , and their total length is  $4\sqrt{50} = 20\sqrt{2}$ . A right rectangular prism with edge lengths 3, 4, and 5 satisfies the conditions of the problem.

15. **Answer (C):** Because  $k \ln k = \ln(k^k)$  and the log of a product is the sum of the logs,  $p = \ln \prod_{k=1}^6 k^k$ . Therefore  $e^p$  is the integer  $1^1 \cdot 2^2 \cdot 3^3 \cdot 4^4 \cdot 5^5 \cdot 6^6 = 2^{16} \cdot 3^9 \cdot 5^5$ , and the largest power of 2 dividing  $e^p$  is  $2^{16}$ .

16. **Answer (E):** Because  $P(0) = k$ , it follows that  $P(x) = ax^3 + bx^2 + cx + k$ . Thus  $P(1) = a + b + c + k = 2k$  and  $P(-1) = -a + b - c + k = 3k$ . Adding these equations gives  $2b = 3k$ . Hence

$$\begin{aligned} P(2) + P(-2) &= (8a + 4b + 2c + k) + (-8a + 4b - 2c + k) \\ &= 8b + 2k = 12k + 2k = 14k. \end{aligned}$$

OR

Let  $(P(-2), P(-1), P(0), P(1), P(2)) = (r, 3k, k, 2k, s)$ . The sequence of first differences of consecutive values is  $(3k - r, -2k, k, s - 2k)$ , the sequence of second differences is  $(r - 5k, 3k, s - 3k)$ , and the sequence of third differences is  $(8k - r, s - 6k)$ . Because  $P$  is a cubic polynomial, the third differences are equal, so  $P(-2) + P(2) = r + s = 14k$ .

17. **Answer (E):** The line passing through point  $Q = (20, 14)$  with slope  $m$  has equation  $y - 14 = m(x - 20)$ . The requested values for  $m$  are those for which the system

$$\begin{cases} y - 14 = m(x - 20) \\ y = x^2 \end{cases}$$

has no solutions. Solving for  $y$  in the first equation and substituting into the second yields  $m(x - 20) + 14 = x^2$ , which reduces to  $x^2 - mx + (20m - 14) = 0$ . This equation has no solution for  $x$  when the discriminant is negative, that is, when  $m^2 - 4 \cdot (20m - 14) = m^2 - 80m + 56 < 0$ . This quadratic in  $m$  is negative between its two roots  $40 \pm \sqrt{40^2 - 56}$ , which are the required values of  $r$  and  $s$ . The requested sum is  $r + s = 2 \cdot 40 = 80$ .

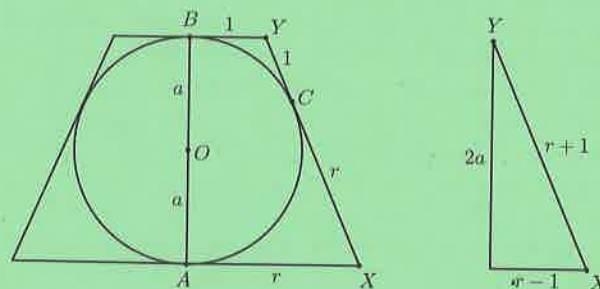
18. **Answer (B):** The circular arrangement 14352 is bad because the sum 6 cannot be achieved with consecutive numbers, and the circular arrangement 23154 is bad because the sum 7 cannot be so achieved. It remains to show that these are the only bad arrangements. Given a circular arrangement, sums 1 through 5 can be achieved with a single number, and if the sum  $n$  can be achieved, then the sum  $15 - n$  can be achieved using the complementary subset. Therefore an arrangement is not bad as long as sums 6 and 7 can be achieved. Suppose 6 cannot be achieved. Then 1 and 5 cannot be adjacent, so by a suitable rotation and/or reflection, the arrangement is  $1bc5e$ . Furthermore,  $\{b, c\}$  cannot equal  $\{2, 3\}$  because  $1 + 2 + 3 = 6$ ; similarly  $\{b, c\}$  cannot equal  $\{2, 4\}$ . It follows that  $e = 2$ , which then forces the arrangement to be 14352 in order to avoid consecutive 213. This arrangement is bad. Next suppose that 7 cannot be achieved. Then 2 and 5 cannot be adjacent, so again without loss of generality the arrangement is  $2bc5e$ . Reasoning as before,  $\{b, c\}$  cannot equal  $\{3, 4\}$  or  $\{1, 4\}$ , so  $e = 4$ , and then  $b = 3$  and  $c = 1$ , to avoid consecutive 421; therefore the arrangement is 23154, which is also bad. Thus there are only two bad arrangements up to rotation and reflection.

19. **Answer (E):** Assume without loss of generality that the radius of the top base of the truncated cone (frustum) is 1. Denote the radius of the bottom base by  $r$  and the radius of the sphere by  $a$ . The figure on the left is a side view of the frustum. Applying the Pythagorean Theorem to the triangle on the right yields  $r = a^2$ . The volume of the frustum is

$$\frac{1}{3}\pi(r^2 + r \cdot 1 + 1^2) \cdot 2a = \frac{1}{3}\pi(a^4 + a^2 + 1) \cdot 2a.$$

Setting this equal to twice the volume of the sphere,  $\frac{4}{3}\pi a^3$ , and simplifying gives  $a^4 - 3a^2 + 1 = 0$ , or  $r^2 - 3r + 1 = 0$ . Therefore  $r = \frac{3+\sqrt{5}}{2}$ .





20. **Answer (B):** The domain of  $\log_{10}(x-40) + \log_{10}(60-x)$  is  $40 < x < 60$ . Within this domain, the inequality  $\log_{10}(x-40) + \log_{10}(60-x) < 2$  is equivalent to each of the following:  $\log_{10}((x-40)(60-x)) < 2$ ,  $(x-40)(60-x) < 10^2 = 100$ ,  $x^2 - 100x + 2500 > 0$ , and  $(x-50)^2 > 0$ . The last inequality is true for all  $x \neq 50$ . Thus the integer solutions to the original inequality are 41, 42, ..., 49, 51, 52, ..., 59, and their number is 18.

21. **Answer (C):** Let  $x = BE = GH = CF$ , and let  $\theta = \angle DHG = \angle AGJ = \angle FKH$ . Note that  $AD = GJ = HK = 1$ . In right triangle  $GDH$ ,  $x \sin \theta = DG = 1 - AG = 1 - \cos \theta$ , so  $x = \frac{1 - \cos \theta}{\sin \theta}$ . Then  $1 = CD = CF + FH + HD = x + \sin \theta + x \cos \theta$ . Substituting for  $x$  gives

$$\begin{aligned} 1 &= \frac{1 - \cos \theta}{\sin \theta} + \sin \theta + \frac{1 - \cos \theta}{\sin \theta} \cdot \cos \theta \\ &= \frac{(1 - \cos \theta)(1 + \cos \theta)}{\sin \theta} + \sin \theta \\ &= \frac{\sin^2 \theta}{\sin \theta} + \sin \theta = 2 \sin \theta. \end{aligned}$$

It follows that  $\sin \theta = \frac{1}{2}$ , so  $\theta = 30^\circ$ , and

$$x = \frac{1 - \frac{\sqrt{3}}{2}}{\frac{1}{2}} = 2 - \sqrt{3}.$$

OR

Let  $a = EK$ ,  $b = EJ$ , and  $c = JK = BE$ . Then triangles  $KEJ$ ,  $GDH$ , and  $JAG$  are similar right triangles and it follows that  $a^2 + b^2 = c^2$ ,  $\frac{a}{c} = 1 - b - c$ , and  $\frac{b}{c} = 1 - a$ . The first equation is equivalent to  $a^2 = (c + b)(c - b)$ , and the last equation is equivalent to  $ac = c - b$ . Multiplying by  $c + b$  and equating to the first equation gives  $ac(c + b) = (c + b)(c - b) = a^2$ . Because  $a > 0$ , it follows that  $a = c(c + b)$ . Plugging into the second equation gives  $c(1 - b - c) = c(c + b)$ . Because  $c > 0$ , it follows that  $c + b = \frac{1}{2}$ . Thus  $a = \frac{c}{2}$  and

$$c^2 = a^2 + b^2 = \frac{c^2}{4} + \left(\frac{1}{2} - c\right)^2.$$

Solving for  $c$  gives  $c = 2 \pm \sqrt{3}$ . If  $c = 2 + \sqrt{3}$ , then  $b = \frac{1}{2} - c = -\frac{3}{2} - \sqrt{3} < 0$ . Thus  $BE = c = 2 - \sqrt{3}$ .

22. **Answer (C):** First note that once the frog is on pad 5, it has probability  $\frac{1}{2}$  of eventually being eaten by the snake, and a probability  $\frac{1}{2}$  of eventually exiting the pond without being eaten. It is therefore necessary only to determine the probability that the frog on pad 1 will reach pad 5 before being eaten.

Consider the frog's jumps in pairs. The frog on pad 1 will advance to pad 3 with probability  $\frac{9}{10} \cdot \frac{8}{10} = \frac{72}{100}$ , will be back at pad 1 with probability  $\frac{9}{10} \cdot \frac{2}{10} = \frac{18}{100}$ , and will retreat to pad 0 and be eaten with probability  $\frac{1}{10}$ . Because the frog will eventually make it to pad 3 or make it to pad 0, the probability that it ultimately makes it to pad 3 is  $\frac{72}{100} \div \left(\frac{72}{100} + \frac{10}{100}\right) = \frac{36}{41}$ , and the probability that it ultimately makes it to pad 0 is  $\frac{10}{100} \div \left(\frac{72}{100} + \frac{10}{100}\right) = \frac{5}{41}$ .

Similarly, in a pair of jumps the frog will advance from pad 3 to pad 5 with probability  $\frac{7}{10} \cdot \frac{6}{10} = \frac{42}{100}$ , will be back at pad 3 with probability  $\frac{7}{10} \cdot \frac{4}{10} + \frac{3}{10} \cdot \frac{8}{10} = \frac{52}{100}$ , and will retreat to pad 1 with probability  $\frac{3}{10} \cdot \frac{2}{10} = \frac{6}{100}$ . Because the frog will ultimately make it to pad 5 or pad 1 from pad 3, the probability that it ultimately makes it to pad 5 is  $\frac{42}{100} \div \left(\frac{42}{100} + \frac{6}{100}\right) = \frac{7}{8}$ , and the probability that it ultimately makes it to pad 1 is  $\frac{6}{100} \div \left(\frac{42}{100} + \frac{6}{100}\right) = \frac{1}{8}$ .

The sequences of pairs of moves by which the frog will advance to pad 5 without being eaten are

$$1 \rightarrow 3 \rightarrow 5, 1 \rightarrow 3 \rightarrow 1 \rightarrow 3 \rightarrow 5, 1 \rightarrow 3 \rightarrow 1 \rightarrow 3 \rightarrow 1 \rightarrow 3 \rightarrow 5,$$

and so on. The sum of the respective probabilities of reaching pad 5 is then

$$\begin{aligned} & \frac{36}{41} \cdot \frac{7}{8} + \frac{36}{41} \cdot \frac{1}{8} \cdot \frac{36}{41} \cdot \frac{7}{8} + \frac{36}{41} \cdot \frac{1}{8} \cdot \frac{36}{41} \cdot \frac{1}{8} \cdot \frac{36}{41} \cdot \frac{7}{8} + \cdots \\ &= \frac{63}{82} \left( 1 + \frac{9}{82} + \left(\frac{9}{82}\right)^2 + \cdots \right) \\ &= \frac{63}{82} \div \left( 1 - \frac{9}{82} \right) \end{aligned}$$

$$= \frac{63}{73}.$$

Therefore the requested probability is  $\frac{1}{2} \cdot \frac{63}{73} = \frac{63}{146}$ .

OR

For  $1 \leq j \leq 5$ , let  $p_j$  be the probability that the frog eventually reaches pad 10 starting at pad  $j$ . By symmetry  $p_5 = \frac{1}{2}$ . For the frog to reach pad 10 starting from pad 4, the frog goes either to pad 3 with probability  $\frac{2}{5}$  or to pad 5 with probability  $\frac{3}{5}$ , and then continues on a successful sequence from either of these pads. Thus  $p_4 = \frac{2}{5}p_3 + \frac{3}{5}p_5 = \frac{2}{5}p_3 + \frac{3}{10}$ . Similarly, to reach pad 10 starting from pad 3, the frog goes either to pad 2 with probability  $\frac{3}{10}$  or to pad 4 with probability  $\frac{7}{10}$ . Thus  $p_3 = \frac{3}{10}p_2 + \frac{7}{10}p_4$ , and substituting from the previous equation for  $p_4$  gives  $p_3 = \frac{5}{12}p_2 + \frac{7}{24}$ . In the same way,  $p_2 = \frac{1}{5}p_1 + \frac{4}{5}p_3$  and after substituting for  $p_3$  gives  $p_2 = \frac{3}{10}p_1 + \frac{7}{20}$ . Lastly, for the frog to escape starting from pad 1, it is necessary for it to get to pad 2 with probability  $\frac{9}{10}$ , and then escape starting from pad 2. Thus  $p_1 = \frac{9}{10}p_2 = \frac{9}{10}\left(\frac{3}{10}p_1 + \frac{7}{20}\right)$ , and solving the equation gives  $p_1 = \frac{63}{146}$ .

**Note:** This type of random process is called a Markov process.

23. **Answer (C):** Let  $n = \binom{2014}{k}$ . Note that  $2016 \cdot 2015 \equiv (-1)(-2) \equiv 2 \pmod{2017}$  and  $2016 \cdot 2015 \cdots (2015 - k) \equiv (-1)(-2) \cdots (-(k+2)) \equiv (-1)^k(k+2)! \pmod{2017}$ . Because  $n \cdot k! \cdot (2014 - k)! = 2014!$ , it follows that

$$\begin{aligned} n \cdot k! \cdot (2014 - k)! \cdot ((2015 - k) \cdots 2015 \cdot 2016) \cdot 2 &\equiv \\ 2014! \cdot 2015 \cdot 2016 \cdot (-1)^k(k+2)! &\pmod{2017}. \end{aligned}$$

Thus

$$2n \cdot k! \cdot 2016! \equiv (-1)^k(k+2)! \cdot 2016! \pmod{2017}.$$

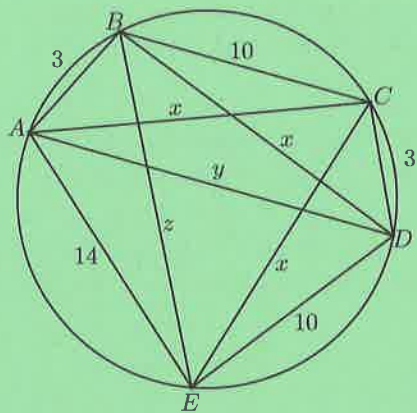
Dividing by  $2016! \cdot k!$ , which is relatively prime to 2017, gives

$$2n \equiv (-1)^k(k+2)(k+1) \pmod{2017}.$$

Thus  $n \equiv (-1)^k \binom{k+2}{2} \pmod{2017}$ . It follows that

$$\begin{aligned} S &\equiv \sum_{k=0}^{62} (-1)^k \binom{k+2}{2} = 1 + \sum_{k=1}^{31} \left( \binom{2k+2}{2} - \binom{2k+1}{2} \right) \\ &= 1 + \sum_{k=1}^{31} (2k+1) = 32^2 = 1024 \pmod{2017}. \end{aligned}$$

24. **Answer (D):** Let  $x = AC$ ,  $y = AD$ , and  $z = BE$ . Because the arcs  $ABC$ ,  $BCD$ , and  $CDE$  are congruent, it follows that  $AC = BD = CE = x$ .



By Ptolemy's Theorem applied to the quadrilaterals  $ABCD$ ,  $ABDE$ , and  $BCDE$ , it follows that

$$10y + 9 = x^2, \quad 30 + 14x = yz, \quad \text{and} \quad 100 + 3z = x^2.$$

Solving for  $y$  and  $z$  in the first and third equations and substituting in the second equation gives

$$30 + 14x = \left(\frac{x^2 - 9}{10}\right) \left(\frac{x^2 - 100}{3}\right) = \frac{x^4 - 109x^2 + 900}{30},$$

which implies that

$$900 + 420x = x^4 - 109x^2 + 900.$$

Thus  $x^3 - 109x - 420 = 0$ . This equation factors as  $(x - 12)(x + 5)(x + 7) = 0$ . Because  $x > 0$  it follows that  $x = 12$ ,  $y = \frac{1}{10}(x^2 - 9) = \frac{135}{10} = \frac{27}{2}$ , and  $z = \frac{1}{3}(x^2 - 100) = \frac{44}{3}$ . The required sum of diagonals equals  $3x + y + z = \frac{385}{6}$ , so  $m + n = 385 + 6 = 391$ .

25. **Answer (D):** If  $x = \frac{1}{2}\pi y$ , then the given equation is equivalent to

$$2 \cos(\pi y) \left( \cos(\pi y) - \cos\left(\frac{4028\pi}{y}\right) \right) = \cos(2\pi y) - 1.$$

Dividing both sides by 2 and using the identity  $\frac{1}{2}(1 - \cos(2\pi y)) = \sin^2(\pi y)$  yields

$$\cos^2(\pi y) - \cos(\pi y) \cos\left(\frac{4028\pi}{y}\right) = \frac{1}{2}(\cos(2\pi y) - 1) = -\sin^2(\pi y).$$

This is equivalent to

$$1 = \cos(\pi y) \cos\left(\frac{4028\pi}{y}\right).$$

Thus either  $\cos(\pi y) = \cos(\frac{4028\pi}{y}) = 1$  or  $\cos(\pi y) = \cos(\frac{4028\pi}{y}) = -1$ . It follows that  $y$  and  $\frac{4028}{y}$  are both integers having the same parity. Therefore  $y$  cannot be odd or a multiple of 4. Finally, let  $y = 2a$  with  $a$  a positive odd divisor of  $4028 = 2^2 \cdot 19 \cdot 53$ , that is  $a \in \{1, 19, 53, 19 \cdot 53\}$ . Then  $\cos(\pi y) = \cos(2a\pi) = 1$  and  $\cos(\frac{4028\pi}{y}) = \cos(\frac{2014\pi}{a}) = 1$ . Therefore the sum of all solutions  $x$  is  $\pi(1 + 19 + 53 + 19 \cdot 53) = \pi(19 + 1)(53 + 1) = 1080\pi$ .

The problems and solutions in this contest were proposed by Bernardo Abrego, Tom Butts, Steven Davis, Peter Gilchrist, Jerry Grossman, Joe Kennedy, Gerald Kraus, Roger Waggoner, Kevin Wang, David Wells, LeRoy Wenstrom, and Ronald Yannone.

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